A Conjecture on Multivariate Polynomial Minimization

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Abstract

We make a conjecture that the number of isolated local minimum points of a $2r$-degree or $(2r+1)$-degree n-variable polynomial is not greater than r^n. We show that this conjecture is the minimal estimate, and is true in several cases. In particular, we show that a cubic polynomial of n variables may have at most one local minimum point though it may have 2^n critical points.

Key words: Multivariate Polynomial Minimization, The Bézout Theorem.

Polynomial Equations

Consider the following system of polynomial equations

$$F(x) = 0, \tag{1}$$

where $F : R^n \to R^n$, $F_i, i = 1, ..., n$, are multivariate polynomials of x.

Such polynomial equations arise from power engineering, robot engineering, chemical engineering and computational chemistry. One hot topic is to find all the (isolated) roots of (1). There are several approaches:

Elimination Method (W.T. Wu, J. Renegar)

Symbolic Manipulation Method (Grobner)

Homotopy Method (T.Y. Li, M. Kojima)

Interval Newton Method (E.R. Hansen)

Branch and Bound Method (C.A. Floudas)

The state of the art is $n = 10 - 20$.

The Bézout Theorem

How many isolated (complex) roots does the polynomial system

$$F(x) = 0 \tag{1}$$

have?

There is a classic theorem on this: The Bézout Theorem. Let the degree of F_i be d_i. Let

$$B := \prod_{i=1}^{n} d_i.$$

B is called the Bézout number of (1). The Bézout Theorem says that the number of isolated complex roots of (1) is not greater than B.

B is only an upper bound. In the 2000 book by D.Y. Cai and F.S. Bai, there are two examples:

1. The Casson-Nogues example: $n = 4, d_1 = 7, d_2 = 8, d_3 = 6, d_4 = 4$,

$$B = 7 \times 8 \times 6 \times 4 = 1344.$$

But the system has only 16 roots.

2. The Matrix Eigenvalue Problem

$$Ax = \lambda x,$$

$$\sum_{i=1}^{n} x^2 = 1,$$

where $x \in R^n$, A is an $n \times n$ matrix. We may regard λ as an additional variable. Then $d_i = 2$ for all i. Hence, $B = 2^{n+1}$ but this system has only $2n$ roots in general.

What is about multivariate polynomial optimization?

Multivariate Polynomial Optimization

Consider

$$\min f(x), \tag{2}$$

where $f : R^n \to R$ is a polynomial of degree d.

Such a multivariate polynomial optimization (MPO) problem arises in engineering problems as well as other optimization problems, such as 0-1 integer linear and quadratic programs, nonconvex quadratic programs and bilinear matrix inequalities. M. Kojima gave a talk "A General Framework for Convex Relaxation of Polynomial Optimization Problems over Cones" in OCA 2002. He calls such a problem a polynomial optimization problem (POP).

Beside M. Kojima, people working on MPO or POP include N. Shor, J.B. Lasserre, etc.

Local Minima

One theoretical question is: How many isolated local minima does MPO (POP)

$$\min f(x) \tag{2}$$

have?

It will be useful to find all the local minima of (2) in the homotopy method for finding a global minimum of (2), as a local minimum may become a global minimum if an additional parameter t is involved. This question in fact is related to Hilbert's 16th problem.

One way is to consider the isolated roots of

$$F(x) = 0, \tag{1}$$

where $F = \nabla f$. By the Bézout Theorem, the number of isolated complex roots of (1) is not greater than

$$B = (d-1)^n. \tag{3}$$

This estimate is too rough, as only real roots of (1) are critical points of (2), and the critical points of (1) include local maxima, local minima and saddle points.

A 1993 Paper

This issue was studied by Durfee, Kronenfeld, Munson, Roy and Westby for $n = 2$ in 1993 in the following paper:

A. Durfee, N. Kronenfeld, H. Munsen, J. Roy and I. Westby, *Counting critical points of real polynomials in two variables*, Amer. Math. Monthly, 100 (1993), 255-271.

They pointed out that this issue is related to Hilbert's 16th problem on the arrangements of ovals of real algebraic curves. Since then, to the best of our knowledge, no further results appear in the literature. This shows that this is a tough issue.

Durfee, Kronenfeld, Munson, Roy and Westby proved that for $n = 2$,

$$N_{\max} + N_{\min} \le \frac{1}{2}d^2 - d + 1,$$

where $N_{\max}$ is the number of local maxima, $N_{\min}$ is the number of local minima. When $d = 3$, their bound is 2. We show that this is true for all n. When $d = 4$, their bound is 5. This is actually the true upper bound, i.e., a two variable quartic polynomial has at most 5 local maxima and local minima. An example is f defined by

$$f(x) = x_1^4 + x_2^4 - 2x_1^2 - 2x_2^2.$$

They also made a conjecture that if $d = 2r$ is even,

$$N_{\max} \le \frac{3}{2}r(r-1) + 1.$$

If we take $d = 4$, i.e., $r = 2$, we have

$$N_{\max} \le 4.$$

If we multiply the polynomial by -1, this implies that

$$N_{\min} \le 4.$$

This is the same as our conjecture.

Our Conjecture

Conjecture (Qi and Teo 2002) A $2r$-degree $2r+1$-degree polynomial of n variables has at most r^n isolated local minima, i.e.,

$$N_{\min} \leq r^n. \tag{4}$$

Note that $B = (d-1)^n = (2r-1)^n$ or $(2r)^n$.

Clearly, this conjecture is true for $n = 1$. This conjecture is also true when f is separable, i.e.,

$$f(x) = \sum_{i=1}^{n} f_i(x_i),$$

where f_i is a polynomial of x_i, and the degree of f_i is not higher than $2r+1$.

A nontrivial case is that $d = 3$, i.e., for cubic polynomials.

Cubic Polynomial

Theorem (Qi and Teo 2002) A cubic polynomial of n variables may have 2^n critical points. However, it has at most and may have one local minimum and one local maximum.

The following example shows that 2^n critical points are achievable. Let f be defined by

$$f(x) = \sum_{i=1}^{n} x_i^3 - 3x_i.$$

Clearly, f has 2^n critical points

$$x = (x_1, \cdots, x_n)^T,$$

where $x_i = 1$ or -1. However, f has only one local minimum

$$x = (1, \cdots, 1)^T,$$

and one local maximum

$$x = (-1, \cdots, -1)^T.$$

Quartic Polynomial

To prove or to disprove our conjecture, the next case is that $d = 4$, i.e., for quartic polynomials.

For $d = 4$, our conjecture is:

$$N_{\min} \leq 2^n.$$

When $n = 2$, our conjecture is the same as the conjecture of Durfee, Kronenfeld, Munson, Roy and Westby, i.e.,

$$N_{\min} \leq 4.$$

As we said before, in this case $(n = 2, d = 4)$, they proved that

$$N_{\max} + N_{\min} \leq 5.$$

An interesting open question is: Can we prove that a quartic two variable polynomial has at most 4 local minima, or construct a quartic two variable polynomial having 5 local minima?