Canonical Dual Transformation Method:

A New Powerful Approach in Global Optimization and Nonconvex Variational Problems

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Duality is a fundamental concept that underlies all most all nature phenomena.
In global optimization, duality theory falls principally into three categories:

(1) the classical saddle Lagrange (minimax) duality in convex problems,
(2) the nice super-Lagrangian bi-duality in geometrically linear systems and
(3) the interesting triality and multi-duality in general nonconvex canonical systems.

The advantages of duality methods and theory rely on a common mathematical structure. Such a structure is independent of the physical contents of the system and exists in wider classes of problems in engineering and sciences.
In convex optimization and variational problems, this structure and associated mathematical theory of duality have been well-studied (see Gao, 2000).
The classical saddle-Lagrangian theory leads to a one-to-one min-max duality theory for primal and dual problems. During the last decade, the so-called *primal-dual interior point method* has emerged as the most important and efficient revolutionary technique in mathematical programming. Actually, the so-call pan-penalty mixed finite element method developed in solid mechanics (see Gao, 1988) can be consider as the *primal-dual exterior point method*, which plays an important role in large scale nonlinear programming. In convex Hamilton systems and D.C. programming (difference of convex functions), the classical Lagrangian associated with the primal problems is usually a so-called super-critical function, i.e. $L(x, y^*)$ is concave in both its variables x and y^*. In this case, a bi-duality theory was proved recently (see Gao, 2000, Chapter 2), i.e. the critical point (x, y^*) of $L(x, y^*)$ could be either minimizer or maximizer of both primal and dual problems.

Duality structure in nonconvex systems and the associated tri-duality theory was originally studied by Gao and Strang for finite deformation problems (Gao-Strang, 1989), where the total potential $P(u) = W(\text{grad } u) - F(u)$ is a nonconvex functional of deformation u, and $F(u)$ is a linear functional (external energy). Since the internal energy $W(D)$ is usually a nonconvex functional of the deformation gradient $D = \text{grad } u$ (a matrix of m by n), the traditional Fenchel-Rockafellar dual transformation leads to a so-called duality gap between the primal problem (minimal potential principle) and the classical dual problem (Levinson complementary variational principle). Based on the duality nature of mechanics (strain and stress relation should be one-to-one), instead of the linear (differential) operator grad, Gao and Strang introduced a quadratic strain measure $E = (\text{grad } u)^T (\text{grad } u)$, which is a symmetric canonical strain tensor, and the associated *complementary gap function*, they were able to recover the duality gap. It turns out that a canonical dual variational problem was obtained (see Gao, 2000, Chapter 6).

Recently, in the study of nonconvex bifurcation problems in engineering mechanics and chaotic dynamical systems, the so-called canonical dual transformation method and associated tri-duality theory were developed (see Gao, 2000a,b). The key idea of this powerful method was from Gao-Strang's original paper, and is based on the following assumption: for any given nonconvex function $P(x)$, by choosing an appropriate operator $N(x)$ (usually nonlinear) such that $P(x)$ can be written in the form of $P(x)= J(x, N(x))$, where $J(x, y)$ is the so-called canonical function, i.e. $J(x, y)$ is either convex or concave in each of its variables. Very often, $P(x)= J(x, N(x))= W(N(x)) - F(x)$.
Since the canonical functions $F(x)$ and $W(y)$ are either convex or concave, their conjugate functions can be easily obtained by solving the following stationary point problem:

$$W^*(y^*) = \text{Sta}_{y} \{ < y , y^* > - W(y)\} .$$

This is the well-known classical Legendre dual transformation. Thus, the extended Lagrangian associated with the canonical dual transformation can be proposed as (see Gao, 2000, Chapter 3):

$$L(x, y^*) = < N(x) , y^* > - W^*(y^*) - F(x).$$

By the Legendre duality, for a given primal variable x, we have

$$P(x) = \text{Sta}_{y^*} L(x, y^*).$$

On the other hand, for a given dual variable y^*, the canonical dual problem can be obtained by solving the following stationary problem:

$$P^d(y^*) = \text{Sta}_{x} L(x, y^*)$$

It is easy to check that if (x, y^*) is a stationary point of L, then

$$P(x) = L(x, y^*)= P^d(y^*).$$

This equality shows that there is no duality gap between the primal and canonical dual problems. When $N(x)$ is a quadratic operator, a very interesting tri-duality theorem was discovered, first in finite deformation theory (Gao, 1997) and then in general global optimization (Gao, 1999). This tri-duality theory reveals the intrinsic symmetry and beauty in nonconvex systems, which can be used to find all local minimizers and local maximizers of $P(x)$.
Very often, N is a m by n matrix with $m < n$.
In this case, the canonical dual transformation method transfer a nonlinear problem in n-dimensional space into a m-dimensional dual problem. This dimension-reduction technique is significantly important in large-scale nonlinear optimization. By the fact that the Legendre conjugate of a nonsmooth function is usually a smooth function in the dual space, the canonical dual transformation method can be used to solve many nonsmooth problems in engineering and science.
Due to the duality of natural phenomena, we know that physical variables appear always in pairs. So for any given nonconvex problem (as long as it comes from reality), the canonical operator $N(x)$ and function W can always be found such that the duality relation (constitutive law) $y^* = DW(y)$ is one-to-one (canonical). Thus, the canonical dual transformation method can be used to solve many very difficult nonsmooth/ nonconvex optimization problems in real world. Recent results (see Gao, 2002) shown that for a large class of nonconvex optimization problems, complete solutions, including

all extremum points (both minimizers and maximizers) and all saddle points, can be obtained by the canonical dual transformation method. This method has been generalized for any nonlinear operator *N(x)* and two sequential canonical dual transformation methods have been proposed (see Gao, 2000, Chapter 4). Extensive applications have been illustrated by concrete real problems in d.c. programming, nonconvex variational inequality, nonsmooth/nonconvex mechanics, phase transitions in physics and material science, non-conservative Hamilton systems and chaotic dynamics . Attached is an encyclopedia article (Duality-Mathematics) for the general reader. Detailed publication list can be found at author's web page: http://www.math.vt.edu/people/gao/papers/paper.html

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