Preface

This book is concerned with topological and differential properties of multivalued mappings and marginal functions. Beside this applications to the sensitivity analysis of optimization problems, in particular nonlinear programming problems with perturbations, are studied. The elaborated methods are primarily obtained by theories and concepts of two former Soviet Union researchers, Demyanov and Rubinov. Consequently, a significant part of the presented results have never been published in English before. Based on the use of directional derivatives as a key tool in studying nonsmooth functions and multifunctions, these results can be considered as a further development of quasidifferential calculus created by Demyanov and Rubinov.

In contrast to other research in this field, especially the recent publication by Bonnans and Shapiro, this book analyses properties of marginal functions associated with optimization problems under quite general constraints defined by means of multivalued mappings. A unified approach to directional differentiability of functions and multifunctions forms the base of the volume.

Different notions of the derivative of a multivalued mapping play an important role in the investigation of a large number of problems. Various kinds of derivatives of multivalued mappings were used in studying economic problems (Rubinov, Aubin), constrained minimax problems (Demyanov), control problems described by differential inclusions (Clarke, Pshenichny, Aubin, Kurzhanski, Frankowska, Mordukhovich and others) and differential games (Krasovski, Subbotin). Moreover, a great field of research is in sensitivity and stability analysis of nonlinear programming problems under perturbations (Aubin, Rockafellar, Ioffe, Shapiro, Bonnans, Auslender, Cominetti and others). Here, as in other works, the weakened and non-unique nature of such an important concept as derivative and differentiability in general, is an expression of one of the most essential features of multivalued analysis in comparison with classical analysis.

In the present book the authors develop a theory of directional differential calculus of multivalued mappings that includes the calculus of higher-order directional derivatives. This approach differs from the one brought in by Bonnans and Shapiro. Its origin goes back to a publication by Demyanov in 1974, where he proposes the concept of tangent (feasible) directions playing the role of the directional derivative of a multivalued mapping. In general, multivalued mappings are not directionally differentiable in the sense described in the book. Therefore, the authors pay a lot of attention to the specification of classes of multivalued mappings being directionally differentiable. These results have not yet been published in English.

In studying the properties of multivalued mappings the fundamental idea of this volume is to use the close relationship between these mappings and their so-called marginal or optimal value functions (extreme value functions defined on the values of multivalued mappings). In particular, differential (as well as topological) properties of multivalued mappings can completely be derived from associated properties of marginal functions and vice versa. Therefore, under quite general assumptions concerning multivalued mappings, such properties as upper and lower semicontinuity, (Lipschitz) continuity or directional differentiability are equivalent to their counterpart in marginal functions theory, in particular, to properties of the distance function being the simplest marginal function.

In marginal functions theory a wide range of results is accessible. The most precise approximations of nonsmooth functions, for example, are accomplished by directional derivatives (provided they exist). First results dealing with calculation of directional derivatives for marginal functions were obtained in linear programming (Mills). Further requirements to study this subject emerged from minimax theory and perturbation analysis of mathematical programming problems. Beginning with the seventies of the last century, research in differential properties of marginal functions started to achieve even more attention (see e. g. Golshtein, Rockafellar, Rubinov, Hiriart-Urruty, Shapiro, Ioffe, Auslender and many others). Examples here are the study of second-order directional derivatives of marginal functions initiated by Demyanov and the second-order derivatives of a more general type introduced by Ben-Tal and Zowe. The study of their existence and constructive computation was topic of recent papers by Auslender, Cominetti, Ioffe, Shapiro, Bonnans and others. Notable in this context is that marginal functions are not necessarily differentiable. Thus, in many papers the trend appeared to specify certain classes of problems (e. g. convex or regular) with at least directionally differentiable marginal functions.

The description and stability analysis of extremum problems with the help of multivalued mappings is helpful in different manner. The description allows not only to obtain very natural assumptions and general results but often also to get considerable advantages in deriving these results. The same is true for the stability analysis of extremum problems with respect to perturbations, where the central problem is the study of differential properties of marginal functions. Moreover, generalized derivatives of marginal functions and their estimates (approximations) can be quite useful for sensitivity analysis of perturbations in extremum problems.

Divided into 5 chapters, the book starts with basic concepts and problems of convex and nonsmooth analysis, which are required as background knowledge for the further treatment.

The second chapter describes topological and differential properties of multivalued mappings. In the first part topological notions, such as uniform boundedness, upper and lower semicontinuity and continuity, Lipschitz and pseudo-Lipschitz properties of mappings and their marginal functions are introduced. The mutual connection between them is established. In the second part different concepts and properties of differentiability and approximation techniques for multivalued mappings are considered. The description of derivatives of mappings in terms of the distance function is obtained. The important lemma about the removal of constraints is proved.

Chapter 3 is devoted to subdifferentials of marginal functions. Estimates of the Clarke subdifferential of an arbitrary marginal function are obtained under quite general assumptions. Moreover, for locally convex optimization problems an exact formula for the calculation of the subdifferential of their marginal functions is obtained as a consequence of the general method proposed above. For the important class of quasidifferentiable functions in the sense of Demyanov and Rubinov basic knowledge (quasidifferential calculus) is given and optimum conditions are described.

In the next chapter a number of general theorems concerning the existence of the directional derivative of the maximum function are proved. These results enable to specify broad classes of directionally differentiable mappings for which we succeed in describing their directional derivatives as well as the directional derivatives of their marginal functions in a constructive way. So-called strongly differentiable multivalued mappings introduced earlier by Tyurin, Banks and Jacobs are studied in detail. The obtained results related to such mappings reveal their connection with other classes of differentiable mappings. Furthermore, a theorem about the directional differentiability of their marginal functions is established.

Chapter 5 is completely devoted to the study of nonlinear mathematical programming problems. A survey of regularity conditions is given at the beginning of the chapter. Certain regularity conditions (linear independence of gradients, Mangasarian-Fromowitz constraint qualification, (R)-regularity condition) are treated and generalized. Special attention will be paid to the study of interdependence between different regularity conditions. A second part of the fifth chapter is concerned with nonlinear programming problems involving perturbations and contains results on the Lipschitz or Hölder behaviour. Differentiability properties of their optimal solutions, upper and lower estimates of Dini derivatives as well as estimates for the Clarke subdifferential of marginal functions are given. Various conditions for the existence of first- and second-order directional derivatives of optimal value functions are described. Quasidifferentiable problems in nonlinear programming are intensively studied and, again, estimates for the potential directional derivative are given.

It remains to say that the book contains results of research work carried out by the authors in recent years in collaboration with colleagues and students of the mathematical departments of the Byelorussian State University of Informatics and Radioelectronics and the Chemnitz University of Technology. Known results from literature are also taken into account (see Bibliographical Comments). We want to note, however, and this was indeed not our object that not all questions connected with this area of research have been covered.

The book is intended for students and experts of mathematics specializing in optimization. It is based on lectures held at Byelorussian State University of Informatics and Radioelectronics, Byelorussia, and Chemnitz University of Technology, Germany. A number of examples have been included in the text for a better understanding of the material.

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The authors would be very grateful to readers who draw their attention to errors or obscurities in the book or suggest any improvements.

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