Duality Principles in Nonconvex Systems

Theory, Methods and Applications

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Preface

The Dao that can be told is not the constant Dao;
The name that can be named is not the real name.
Nameless, the beginning of the universe.
Nameable, the mother of the myriad creatures.
Lao Zi, 1.1

As far as the laws of mathematics refer to reality,
they are not certain;
As far as they are certain,
they do not refer to reality.
Albert Einstein

The term duality as used in our daily life means the sort of harmony of two opposite or complementary parts through which they integrate into a whole. Symmetry and inner beauty in natural phenomena are bound up with duality and, in particular, are significant in art and science. Mathematics lies at the root of duality. The aim of this book is to give, within a unified framework, a self-contained comprehensive presentation of the mathematical theory of duality for general non-convex and non-smooth systems, and the author hopes that it will provide a smooth, stimulating and provocative blend of different flavors of mechanics.

Duality has always been a rich source of inspiration for human knowledge through the centuries. As a philosophy duality had been discussed extensively from the ancient time of great thinkers such as Lao Zi, Chuang Zi, Plato and Heraclitus to the giants of modern physics such as Mach, Maxwell, Einstein and Bohr. In science, duality and mechanics have been complementary partners since the time of the founding masters, Newton, Euler, Lagrange and Gauss, and their subsequent developers such as Hamilton, Legendre, Riemann, Kelvin, Noether, Poincar\'{e}, Hilbert, Cartan, Weyl, and von Neumann. In the present day, the theory of duality has become a vast subject, especially due to the modern work in optimization, game theory, economic science, theoretical physics and chemistry, mathematical programming, variational analysis, nonconvex-nonsmooth analysis and control, critical point theory and in many other areas. However, the duality gap between the two partners is getting larger and larger artificially.

Duality in mathematics, roughly speaking, is a fundamental concept that underlies many aspects of extremum principles in natural systems. Eigenvectors, geodesics, minimal surfaces, KKT conditions, harmonic maps, subharmonics of Hamiltonian systems and equilibrium states of many field equations are all critical points of certain functionals on some appropriate constraint sets or manifolds. For convex (static or Hamiltonian) systems, the mathematical theory of duality is well established due to the existence of a common symmetric framework for different problems. Several books have already dealt with major theoretical components of the subject.

In applications, the so-called primal-dual algorithms have emerged as the most important and useful algorithms in linear programming during this decade. However, the nice symmetry is broken for nonconvex systems, where the dual formulation and the accompanying theory for each problem depend mainly on the intrinsic constitutive law. To develop a simple and correct duality theory usually requires certain ``physical knowledge'' of the system. Many problems in nonconvex systems have remained obscure and open for a long period.

Actually, at the beginning of this century, duality theory and methods for nonconvex problems were studied by engineers and scientists in mechanics. The well-known Hellinger-Reissner complementary energy principle in nonlinear elasticity, proposed by Hellinger in 1914, might be the first dual formulation for nonconvex variational problems. This principle has many important consequences in large deformation theory and computational mechanics. Unfortunately, this very important complementary variational principle is not known by many mathematicians. It turns out that the extremality property of this principle has been an open problem for more than forty years, and this raised many arguments in finite deformation theory and nonconvex mechanics. One of the main results in this book provides the solution of this problem, and leads to an interesting triality theory for nonconvex systems.

It is intended that the present book will fill the (duality) gap between the mathematical and engineering sciences by providing a systematic exposition of duality theory as a self-contained theoretical system with substantial applications in physics, continuum mechanics, large deformation structures, mathematical programming, partial differential equations and geometry.

Most parts of this book contain material that is new, both in its manner of presentation and in its research development. The new results arose naturally during the writing of this book, and it turns out that the development of the project has thereby been vastly enriched.

This is a book of motivated mathematics, i.e. a book of mathematics motivated by duality in natural phenomena, with particular emphasis on the mechanics rather than a book of mathematical analysis, proofs and applications. The book is therefore written in a dual way: by using two very simple examples, involving an elastostatic string and the dynamics of a particle, as a pair of seeds, the theory is grown continuously from the classical mono-duality of one-dimensional convex static equilibria, the nice bi-duality in dynamical systems, through the interesting tri-duality in nonconvex problems to the complicated multi-duality in general canonical systems. A potentially powerful sequential dual canonical transformation method is developed heuristically and illustrated by use of many interesting simple examples as well as comprehensive applications in three-dimensional finite deformation theory. The book divides naturally into three closely interconnected parts with a total of seven chapters. Each chapter provides some motivation, both at the beginning and throughout, and concludes with substantial applications and commentaries which furnish credits and references.

Duality in nature is amazingly beautiful, for it is the way nature was created.
Duality in nature is simply mysterious, for it is the way that nature exists.
It is beautiful because all things were originally created in a splendid harmonious world.
It is mysterious because different creatures have different patterns of duality.
If we are not confused very often about the duality of natural phenomena,
we do not really understand what it is.
This may be the way that we exist.

Serious study of duality theory or any other subject in science is very difficult (and dangerous).
Both Lao Chi and Einstein told us that once a phenomenon has been either named or explained, certain reality has been lost. So, for example, there was a Copenhagen fog in quantum mechanics. Contrary to popular opinion, J. P. Aubin (1993) warned us that ``mathematics is not simply a richer or more precise Language''. About fifty years ago, G\"{o}del's incompleteness theorem destroyed the existing foundations of formal logic and axiomatic systems. The only reason that the author dares to publish this book is that he does not really know what duality is. Thus, this is a book of constructive mathematics, i.e. a book using formal language and duality methods to model natural phenomena approximately, to construct intrinsic frameworks in different fields and to provide ideas, concepts and certain tools for describing natural phenomena and for solving real problems arising in engineering and science.

The functional spaces and operators used in this book are mainly for the convenience and clarity of the descriptions. All these abstract notations are illustrated sufficiently by very simple examples. Many mathematical definitions, theoretical results and numerical methods are explained with either geometrical illustrations or physical applications. With the aid of these potentially useful notations, common mathematical frameworks are constructed for many different systems, independently of their physical content, and the frameworks also apply to wider classes of problems in engineering and science. These common structures are due to certain conservation laws that govern the systems, while the intrinsic symmetries in the abstract frameworks lead to a variety of dualities, each having a wealth of significant and substantial applications. To the author's best knowledge, there are at the present time several other approaches to duality theory in nonconvex analysis that have proved successful in dealing with the difficulties of infinite-dimensional critical point theory (cf., e.g., Ghoussoub, 1993). However, these nice theories require a more substantial background in abstract analysis, algebraic topology and geometry than is required for the elementary approach adopted in this book. The author apologizes for this and to the many people whose contributions to duality theory are not included or even mentioned in this very limited book. But he sincerely hopes to receive both positive and negative feedback from readers in order to stimulate communication and possible cooperation.

The basic prerequisites for the book are multi-variable calculus and linear algebra. Topics on convex analysis and nonsmooth calculus needed for mathematical duality theories are included in one essentially independent chapter. A three-part appendix provides some necessary background from elementary functional analysis. The whole book or selected chapters can be used as a text for upper-undergraduate or graduate courses in variational methods, global optimization, applied mathematics, general engineering and science, mathematical physics and operations research. It can also be used for an advanced course in theoretical mechanics or to supplement courses on nonlinear optimization and nonconvex analysis. It is the author's hope that this book will also serve as a catalyst for the further development, both in theory and in applications, of this fascinating and fertile scientific field.

DYG, September, 1999

Notes:

Lao Zi, an ancient, mystic Chinese philosopher in the fourth century BC. His book Dao De Chin was the first poetical treatise on the Ying-Yang duality. This book, as well as the books by his follower Chuang Chi and by his duality partner Confucius (551-479, BC) have influenced Chinese thought throughout the ages out of all proportion to their lengths.

According to the Shih Ji (Records of the History, the earliest general history of China, written at the beginning of the first century BC by Shi-Ma Chien),

Confucius once visited
him and asked to be instructed in the ``Dao".
Hence always rid yourself of desires in order to observe its secrets;
But always allow yourself to have desires in order to observe its manifestations.
These two are the same
But diverge in name as they issue forth.
Being the same they are called mysteries,
Mystery upon mystery -
The gateway of the manifold secrets.
Lao Zi, Dao De Jing 1.3
Motivated mathematicians must possess a sound knowledge of another discipline and
have an adequate arsenal of mathematical techniques at their fingertips together with
the capacity to create new techniques (often similar to those they already know).
Jean-Pierre Aubin, 1993

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