SERIES EXPANSIONS OF ANALYTICALLY PERTURBED MATHEMATICAL PROGRAMS

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The main motivation for the line of research outlined below can be stated as follows: as long as we have to work with problems that involve parameters that were either measured or estimated, we have to deal with their perturbations. The term *perturbation* signifies that our parameters deviate from nominal values. As a rule, we do not know the value of this deviation. However, we do expect that the deviation is "small".

Frequently, parameters of interest will be coefficients of a matrix. Therefore, it is natural to begin investigations by analysing matrices with perturbed elements. This general idea can be written mathematically in the form of the following *additive* perturbation

$$\tilde{A} = A + D, \tag{1}$$

where A is a matrix of nominal coefficient values, $\tilde{A}$ is a matrix of perturbed data and D is the perturbation itself. We shall distinguish between *regular* and *singular* perturbations. If the rank of the perturbed parameter matrix $\tilde{A}$ in (1) is different from the rank of the original matrix A, then the perturbation is said to be *singular*, otherwise the perturbation is said to be *regular*. Perhaps, the first comprehensive study of the analytic perturbation theory, where eigenvalues of $\tilde{A}$ are analytic functions of a perturbation parameter ε, was given in the book of Kato [15].

However, the main focus of the present line of research is not the above perturbed operator equation but rather a range of applications in the context of perturbed optimisation problems. It should be noted that sensitivity analysis of mathematical programs is a research subject in which there is substantial amount of activity, internationally. This is evidenced by a sustained sequence of books and papers devoted to this topic almost since its inception (see, for instance Dantzig [7], Fiacco [9], Gaitsgory and Pervozvanskii [20], Gal [11], Gal and Greenberg [12] and Levitin [19]). However, the present approach has been inspired by Jeroslow [13], [14] and Eaves and Rothblum [8].

In particular, we propose to describe the asymptotic behaviour of solutions[1] to a generic, perturbed, mathematical program[2]:

$$\begin{array}{lll} & \max f(\mathbf{x},\varepsilon) & \\ \text{s.t.} & \text{(i)} \quad g_i(\mathbf{x},\varepsilon) = 0;\ i = 1,\cdots,m & \\ & \text{(ii)} \quad -h_j(\mathbf{x},\varepsilon) \geq 0;\ j = 1,\cdots,p, & \end{array} \tag{MP(ε)}$$

where $\mathbf{x} \in \mathbb{R}^n, \varepsilon \in [0,\infty)$ and f, g_i's, h_j's are functions on $\mathbb{R}^n \times [0,\infty)$. The case $\varepsilon = 0$ corresponds to the underlying *unperturbed program* that will be denoted by (MP(0)). The parameter, ε, will be called the *perturbation*. We are especially concerned with characterising solutions, $\mathbf{x}^*(\varepsilon)$, of (MP($\varepsilon$)) as functions of the perturbation parameter, ε, and in their limiting behaviour as $\varepsilon \downarrow 0$. We consider the (MP(ε)) problem at three levels of generality.

[1]The word "solution" is used in a broad sense, at this stage. In some cases the solution will, indeed, be a global optimum while in other cases it will be only a local optimum or a stationary point.

[2]Clearly, the theory for minimization parallels that for maximization

A. Asymptotic Linear Programming. Here all functions are linear in $\mathbf{x}$ and the problem (MP(ε)) can be converted to an essentially equivalent perturbed linear program:

$$\begin{aligned} &\max[\mathbf{c}^T(\varepsilon)\mathbf{x}] \\ \text{s.t.}\quad &A(\varepsilon)\mathbf{x} = \mathbf{b}(\varepsilon) \\ &\mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{LP(ε)}$$

B. Asymptotic Polynomial Programming. Here all the functions $f(\mathbf{x},\varepsilon), g_i(\mathbf{x},\varepsilon)$ and $h_j(\mathbf{x},\varepsilon)$ are polynomials in $\mathbf{x}$ and ε.

C. Asymptotic Analytic Programming. Here all the functions $f(\mathbf{x},\varepsilon), g_i(\mathbf{x},\varepsilon)$ and $h_j(\mathbf{x},\varepsilon)$ are analytic functions in $\mathbf{x}$ and ε.

In the order of decreasing complexity (and generality), recent developments in this line of research are as follows:

1. It was demonstrated (see [6]) that under mild conditions there exists a neighbourhood $(0,\varepsilon^*)$, an integer $M > 0$ and a solution $\mathbf{x}^*(\varepsilon)$ of (MP(ε)) in that neighbourhood that is expressible as a Puiseux series of the form

$$\mathbf{x}^*(\varepsilon) = \sum_{\nu=K}^{\infty} \varepsilon^{\frac{\nu}{M}} \mathbf{c}_\nu. \tag{PS}$$

The analysis invokes the theory of complex analytic varieties (e.g. see Whitney [21]) to characterize the solution sets of Karush-Kuhn-Tucker equations, in the complex domain (of several complex variables) in order to understand the structure of the "real part" of these sets.

To see how fractional powers present in (PS) can naturally arise in mathematical programming consider the following simple example.

Example 1: Let

$$f(x_1,x_2,\varepsilon) = \frac{x_1^4}{4} + \frac{x_2^4}{4} + \frac{\varepsilon}{3}x_1^3x_2 + \varepsilon x_1$$

and consider its stationary points satisfying

$$\frac{\partial f}{\partial x_1} = x_1^3 + \varepsilon x_1^2 x_2 + \varepsilon = 0; \quad \frac{\partial f}{\partial x_2} = x_2^3 + \frac{\varepsilon}{3}x_1^3 = 0.$$

It is easy to check that the solutions $(x_1(\varepsilon), x_2(\varepsilon))$ satisfy $x_2(\varepsilon) = -[x_1(\varepsilon)\varepsilon^{\frac{1}{3}}]/\sqrt[3]{3}$ and $x_1^3(\varepsilon)[1 - \varepsilon^{\frac{4}{3}}/\sqrt[3]{3}] = -\varepsilon$ and hence that

$$x_1(\varepsilon) = -\varepsilon^{\frac{1}{3}} - \varepsilon^{\frac{5}{3}}/3\sqrt[3]{3}\cdots; \quad x_2(\varepsilon) = -\varepsilon^{\frac{2}{3}}/\sqrt[3]{3} + \varepsilon^2/3\sqrt[3]{9}\cdots.$$

Since $(x_1(\varepsilon), x_2(\varepsilon)) \xrightarrow{\varepsilon\downarrow 0} (x_1(0), x_2(0))$, the above solution is well behaved, despite the presence of fractional powers in the series expansion.

The above suggests that the understanding of the expansion (PS) is – in many cases – the key to the understanding of the asymptotic behaviour of solutions to the mathematical program (MP(ε)). Indeed, this approach promises to offer a unified analytic perspective of quite a diverse range of asymptotic behaviours.

2. In [3] it is shown that - in the case B, above - it is possible to discriminate between the cases $M > 1$ and $M = 1$ where (PS) reduces to a Laurent series (LS), as well as between the cases $K < 0$ and $K \geq 0$ where (PS) reduces to a Taylor series (TS). This was done with the help of Buchberger's elimination algorithm from the Gröbner bases theory by replacing the defining Kuhn-Tucker equations of the variety by its Gröbner basis, that has the property that one of its elements is bivariate, i.e. is a polynomial in ε and one of the decision variables only. The work [4] on application of the Gröbner bases to multivariate perturbations of polynomial mathematical programs is in progress.

3. In the case (A), in [5] it was shown that algorithmic procedures can be easily developed. In this case it is sufficient to restrict consideration to the field of Laurent series and it is possible to invoke a recursive updating scheme developed in [1] and [2] to effectively pivot from one basic feasible solution to another until an asymptotically optimal basic feasible solution of (LP(ε)) is found. This method was named the "Asymptotic Simplex Method". See [17] and [18] for related earlier works.

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